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Parametric spectral statistics in unitary random matrix ensembles: from distribution functions to intra-level correlations

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Abstract

We establish a general framework to explore parametric statistics of individual energy levels in unitary random matrix ensembles. For a generic confinement potential $\text{Tr } W(H)$, we (i) find the joint distribution functions of the eigenvalues of H and $H' = H + V$ for an arbitrary fixed V both for finite matrix size N and in the ‘thermodynamic’ $N \rightarrow \infty$ limit; (ii) derive many-point parametric correlation functions of the two sets of eigenvalues and show that they are naturally parametrized by the eigenvalues of the *reactance matrix* for scattering off the ‘potential’ V ; (iii) prove the universality of the correlation functions in unitary ensembles with non-Gaussian non-invariant confinement potential $\text{Tr } W(H - V)$; (iv) establish a general scheme for exact calculation of *level-number-dependent* parametric correlation functions and apply the scheme to the calculation of *intra-level* velocity autocorrelation function and the distribution of parametric level shifts.

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1. Introduction

Parameter-dependent evolution of spectra in random matrix ensembles has been a subject of investigation almost from the inception of the random matrix theory (RMT) [1, 2]. Dyson [1] showed that the joint probability distribution function of the eigenvalues in the classic random matrix ensembles can be alternatively obtained as an equilibrium distribution of a one-dimensional gas of classical charges that undergo Brownian motion in fictitious time while simultaneously logarithmically repelling each other. The same dynamics of eigenvalues arises if the corresponding random matrices themselves execute Brownian motion [3] according to a sequence of mappings $H \mapsto H + dV$ where H is drawn from a random matrix ensemble and

dV is the elementary step of the random walk *in the space of matrices* of the same symmetry as H .

However, despite the theoretical interest in these types of models, the experimental motivation in the early days of RMT was lacking since the nuclear spectra for which RMT was originally devised [4] are essentially immutable. Later, the range of RMT applications was broadened to include systems such as mesoscopic quantum structures (e.g., quantum dots) which can be subject to a variety of external perturbations [5]. In fact, controlled external perturbations such as magnetic field or gate potential often serve as important experimental tools to study statistical properties of such systems. Thus, instead of a sequence of random steps dV one is led to consider finite mappings $H \mapsto H' = H + V$, where V is a fixed matrix with the same symmetry as the ensemble from which the matrices H are drawn¹ [6–9].

While magnetic field clearly belongs to the class of extended perturbations, in many applications the assumption that the rank of V scales with N may not prove to be adequate in the RMT modelling of mesoscopic devices. External perturbations such as an STM tip, a change in the potential on a finger-shaped gate or a jump in the position of a bi- or multi-stable defect represent potentials that are localized in space. In the RMT language, localized potentials are modelled by matrices V of finite rank r . Such matrices can be represented using a finite set of orthonormal complex vectors $\{\mathbf{a}_\kappa\}_{\kappa=1}^r$ as $V = N \sum_{\kappa=1}^r v_\kappa \mathbf{a}_\kappa \otimes \mathbf{a}_\kappa^\dagger$. (For technical reasons related to the use of the Itzykson–Zuber–Harish-Chandra (IZHC) integral [10], our consideration here is restricted to the unitary ensemble, hence the requirement that \mathbf{a}_κ are complex.)

In the standard RMT one can distinguish two classes of correlation functions between the levels [2]. In the first class the levels are ‘labelled’ by their positions (‘energies’ in the Hamiltonian interpretation of the random matrices), so that the fundamental object is the many-point correlation function of the density of states (DoS) $\nu(E) = \sum_{\alpha=1}^N \delta(E - \epsilon_\alpha)$,

$$\tilde{R}_n(\{E\}_{i=1}^n) = \left\langle \prod_{i=1}^n \nu(E_i) \right\rangle \quad (1)$$

where the angular brackets denote averaging over the matrix ensemble. In the bulk region of the spectrum the correlation functions are translationally invariant with respect to the simultaneous shifts of all energies, so that \tilde{R}_n is actually a function of $n - 1$ energy differences. Such correlation functions are known to have determinantal structures in all three classic Wigner–Dyson ensembles. For example, in the unitary ensemble, the $N \rightarrow \infty$ limit results in

$$\tilde{R}_n(\{E\}_{i=1}^n) = \det_{ij} \tilde{k}(\epsilon_i - \epsilon_j) \quad (2)$$

where $\tilde{k}(\epsilon_i - \epsilon_j) = k(\epsilon_i - \epsilon_j) - \delta(\epsilon_i - \epsilon_j)\theta(i - j)$, $k(u) = \sin \pi u / \pi u$ is the celebrated sine kernel of the Wigner–Dyson RMT, the dimensionless energies $\epsilon \equiv E/\bar{\Delta}$ are normalized by the mean level spacing $\bar{\Delta} = \langle \nu(E) \rangle^{-1} \equiv \rho(E)^{-1}$, and the step-function θ is regularized so that $\theta(0) = 0$. The term involving the δ -function accounts for the self-correlation of levels. Analogous structures with more involved kernels arise in other ensembles. While originally derived in the context of the Gaussian unitary ensemble (GUE), the correlation kernel k acquires the same universal form in the bulk scaling limit for a wide class of non-Gaussian random matrix ensembles [11]

$$\mathcal{P}(H) = C_U e^{-\text{Tr } W(H)} \quad (3)$$

where $W(H)$ is an even polynomial, and C_U is the normalization constant.

¹ A separate subject which is beyond the scope of this paper concerns mappings such that the symmetry of V is lower than the symmetry of H .

In the second class of correlation functions the levels are labelled by their *numbers*, while their energies may or may not enter as additional variables depending on the specific correlation function. An archetypal example is the distribution of level spacings $\mathfrak{S}_1(\omega)$ which is a particular case of the distribution $\mathfrak{S}_q(\omega)$ of distances between a level α and another level separated from α by $q - 1$ other levels:

$$\mathfrak{S}_q(\omega) = \bar{\Delta} \langle \delta(\epsilon_\alpha - E) \delta(\epsilon_{\alpha+q} - E - \omega) \rangle. \tag{4}$$

The distribution function \mathfrak{S}_1 is actually a *correlation* function between an arbitrary level α and its *neighbour* $\alpha + 1$. Translational invariance in the bulk of the spectrum ensures that this correlation function does not depend on either E or α . In contrast with (2) no summation over α is involved, so that level-number-specific information is retained, and, indeed, near the edge of the spectrum the same correlation function—now as a function of both level numbers—describes the distribution of distances between, say, the second lowest and the third lowest levels.

The two types of the correlation functions are related to each other *via* infinite series. Thus, an equivalent expression for $\mathfrak{S}_1(\omega)$ without referring to specific level numbers is the joint probability to find two levels at positions E and $E + \omega$ and *all other levels* in the interval $\mathbb{R}/[E, E + \omega]$. After straightforward algebra [2], $\mathfrak{S}_1(\omega)$ is written as

$$\mathfrak{S}_1(\omega) = -\bar{\Delta} \partial_\omega^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_E^{E+\omega} dE_1 \dots \int_E^{E+\omega} dE_n R_n(\{E_i\}_{i=1}^n) \tag{5}$$

where $R_n(\{E\}_{i=1}^n) = \det_{ij} k(E_i - E_j)$ is the part of \tilde{R}_n from which the level self-correlation terms are excluded. Combined with the determinantal structure of R_n , this formula allows one to express \mathfrak{S}_1 in terms of the derivative of a certain generating function, $\mathfrak{S}_1 = -\bar{\Delta} \partial_\omega^2 \det_{[E, E+\omega]}(\mathbb{I} - \hat{k})$, where the determinant is understood as a Fredholm determinant on the space of functions with the support in the interval $[E, E + \omega]$, and the kernel \hat{k} is defined through its matrix elements: $\hat{k}(u, w) = k(u - w)$. Analogous expressions can be obtained for other \mathfrak{S}_q . The Fredholm determinant itself, the generating function, is the probability to find *no* levels in the interval $[E, E + \omega]$. Generic \mathfrak{S}_q is similarly related to the Fredholm determinant expression for the probability to find $q - 1$ levels in a given interval.

Conversely, the correlation function $R_2(E, E + \omega)$ can be expressed as an infinite sum over \mathfrak{S}_q ,

$$R_2(E, E + \omega) = \sum_{q=-\infty}^{\infty} \mathfrak{S}_q(\omega) \tag{6}$$

which is a simple restatement of the fact that the conditional (on the existence of a level α at E) probability to find *any* level at $E + \omega$ is the sum over q of the (conditional) probabilities to find the $(\alpha + q)$ th level at that position. Operationally, however, R_n is usually more readily accessible than \mathfrak{S}_q , so that equation (6) and its analogues are not often used in practical calculations.

In direct analogy with the duality between the two classes of correlation functions described above for ordinary RMT, one can introduce their generalizations in the parametric RMT. Thus, instead of R_n we are now led to define the parametric many-point correlation function as

$$R_{nm}(\{E_i\}_{i=1}^n, \{E'_j\}_{j=1}^m) = \left\langle \prod_{i=1}^n v(E_i) \prod_{j=1}^m v'(E'_j) \right\rangle \tag{7}$$

where $v'(E) = \text{Tr} \delta(E - H')$.

Except in the $r = 1$ case (see below), the ‘level spacing’ in the combined set $\{E\} \cup \{E'\}$ does not have an immediate physical interpretation. A more meaningful analogue of equation (4) is the distribution of distances between a level α and the parametric ‘descendant’ of the level $\alpha + q$,

$$P_q(\omega, \mathfrak{x}) = \bar{\Delta} \langle \delta(\epsilon_\alpha - E) \delta(\epsilon_{\alpha+q}(\mathfrak{x}) - E - \omega) \rangle \quad (8)$$

where \mathfrak{x} is the set of the parameters characterizing V . Due to universality [7, 12], \mathfrak{x} can be identified either with a single parameter x measuring the overall strength of V in the case of extended (infinite rank) perturbations, or with a set of at most r parameters in the finite rank case (see section 3). P_0 is simply the distribution of shifts of a *single* level under the influence of a parametric perturbation. Note the absence of summation over α ; nevertheless, in the bulk of the spectrum P_q is not a function of α .

A related quantity which in the past has been extensively studied in the context of parametric correlations in chaotic and disordered systems using either numerical or approximate analytical techniques [7, 13–16] is the *single level* velocity correlation function $C_0(\mathfrak{x})$. Within the formalism developed in the present paper, it is convenient to represent it as a special case of the following correlation function,

$$C_q(\mathfrak{x}) = \langle \partial_{\mathfrak{x}} \epsilon_\alpha(\mathfrak{x} = 0) \partial_{\mathfrak{x}} \epsilon_{\alpha+q}(\mathfrak{x}) \rangle \quad (9)$$

which represents the correlation of responses to the perturbation of two levels separated by a distance \mathfrak{x} in the parameter space, and by $q - 1$ levels in the level sequence.

The objective of this paper is to present a set of interrelated developments in the parametric RMT. In the next section we construct a formalism to determine the joint distribution functions for the eigenvalues of H and H' for perturbing matrices V of arbitrary rank r . In particular, we concentrate on the case when r is finite in the $N \rightarrow \infty$ limit, although the results are equally applicable in the opposite limit where we reproduce the Gaussian transition kernel implicit in the earlier work on the subject [7, 18].

In section 3.1 we use the joint distribution functions to derive the many-point correlation function R_{nm} . We show that the latter possesses the determinantal structure at arbitrary r [17], generalizing the result previously known only in the context of two-matrix models [7, 18] (i.e., with V itself drawn from a random matrix ensemble), and thus corresponding to $r = N$ in the classification adopted here. In section 3.2 these results are applied to prove (on a physical level of rigour) that the universality of the Wigner–Dyson sine kernel extends to a class of non-invariant ensembles characterized by the distribution function $\mathcal{P}_V(H) = C_U \exp\{-\text{Tr } W(H - V)\}$.

The final objective of this paper is to present (section 3.3) a general framework for the calculation of $P_q(\omega, \mathfrak{x})$, $C_q(\mathfrak{x})$ and related *number-dependent* correlation functions in terms of the corresponding family of generating functions. Owing to the determinantal structure of R_{nm} [7, 17, 18] (see also section 3.1 below), the latter possess representations in terms of Fredholm determinants of certain parameter-dependent integral kernels.

2. Joint distributions of eigenvalues

Consider a set of pairs of matrices H and $H' = H + V$, with H being a random Hermitian $N \times N$ matrix H drawn from the unitary ensemble $\mathcal{P}(H)$ and V is a deterministic matrix of rank r . The joint distribution of the matrices H and H' is

$$\mathcal{P}(H, H') = \mathcal{P}(H) \delta(H + V - H') \quad (10)$$

where the matrix δ -function is understood as the product of ordinary scalar δ -functions—one per each of the N^2 independent variables of H . The joint distribution function of the combined

set of eigenvalues $\{\epsilon_\alpha\}_{\alpha=1}^N$ of the matrix H and $\{\epsilon'_\beta\}_{\beta=1}^N$ of H' is obtained from the above expression by integrating over the angular degrees of freedom,

$$\mathcal{P}(\{\epsilon\}, \{\epsilon'\}) = \int \Delta^2(H)\Delta^2(H') d\mu(U) d\mu(U') \mathcal{P}(H, H') \tag{11}$$

where $d\mu(U)$ is the invariant Haar measure on the unitary group $U(N)$, and $\Delta(H)$ is the Vandermonde determinant,

$$\Delta(H) \equiv \Delta(\{\epsilon\}) = \prod_{1 \leq \alpha < \beta \leq N} (\epsilon_\alpha - \epsilon_\beta). \tag{12}$$

With the help of a Lagrange multiplier matrix λ (it is straightforward to show that independent components of λ can be arranged into a Hermitian matrix) the expression for the joint distribution function takes the form

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) = C_U \int \Delta^2(H)\Delta^2(H')\Delta^2(\lambda) d\mu(U) d\mu(U') \frac{d\mu(U_\lambda)}{(2\pi)^{N^2-N}} \prod_{k=1}^N \frac{d\lambda_k}{2\pi} \\ \times \exp\{i \operatorname{Tr}(\lambda H + \lambda V - \lambda H')\} \exp\{-W(H)\} \end{aligned} \tag{13}$$

where λ_k are the eigenvalues of λ . Integration over the angular degrees of freedom of H and H' can be performed with the help of the IZHC integral [10]

$$\int d\mu(U) \exp\{i \operatorname{Tr} A U B U^\dagger\} = c \frac{\det_{\alpha\beta} [\exp(ia_\alpha b_\beta)]}{\Delta(A)\Delta(B)} \tag{14}$$

where $\{a_\alpha\}_{\alpha=1}^N, \{b_\beta\}_{\beta=1}^N$ are the complete sets of the eigenvalues of the matrices A and B , respectively, and $c = (i/2)^{(N^2-N)/2} \prod_{j=1}^N j!$. The remaining integral thus takes the form

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) = \frac{C_U |c|^2}{(2\pi)^{N^2}} \int \Delta(H)\Delta(H') d\mu(U_\lambda) \prod_{k=1}^N d\lambda_k \exp\{i \operatorname{Tr}(\lambda V)\} \\ \times \sum_{\sigma, \sigma'} (-1)^{\sigma+\sigma'} \exp \left\{ i \sum_{\alpha=1}^N (\epsilon_\alpha \lambda_{\sigma(\alpha)} - \epsilon'_\alpha \lambda_{\sigma'(\alpha)}) \right\} \exp \left\{ - \sum_{\alpha=1}^N W(\epsilon_\alpha) \right\} \end{aligned} \tag{15}$$

where the sum over σ runs over all permutations of the indices $1, \dots, N$, and $(-1)^\sigma$ denotes the signature of the permutation. The ordering of the energy levels in the Vandermonde determinants is assumed to be in the increasing order. Due to the invariance of the measure $\mu(U)$, the integral in (15) does not depend on the ‘angular’ degrees of freedom of V . The latter can thus be thought of as ‘gauge’ degrees of freedom of (15), and integrated over with an appropriate measure $d\mu_{(r)}(\{\mathbf{a}_\kappa\})$. A convenient choice for $d\mu_{(r)}(\{\mathbf{a}_\kappa\})$ is the uniform distribution of all components of \mathbf{a}_κ constrained only by the requirement of orthonormality. U_λ rotations can now be absorbed into the rotations of V , and the integral over $d\mu(U_\lambda)$ gives a constant $\mu_N = \int d\mu(U_\lambda) = (2\pi)^{(N^2-N)/2} / \prod_{j=1}^N j!$. Equation (15) can now be rewritten as

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) = \frac{C_U |c|^2 \mu_N}{(2\pi)^{N^2-N}} \exp \left(- \sum_{\alpha=1}^N W(\epsilon_\alpha) \right) \Delta(H)\Delta(H') \\ \times \int d\mu_{(r)}(\{\mathbf{a}_\kappa\}) \det_{\alpha\beta} \delta(\epsilon_\alpha - \epsilon'_\beta - V_{\alpha\alpha}) \end{aligned} \tag{16}$$

where $V_{\alpha\alpha}$ are the diagonal elements of $V = N \sum_{\kappa=1}^r v_\kappa \mathbf{a}_\kappa \otimes \mathbf{a}_\kappa^\dagger$. It is worth noting at this point that if integrating over $d\mu_{(r)}(\{\mathbf{a}_\kappa\})$ does not introduce correlations between $V_{\alpha\alpha}$, the distribution function acquires, apart from the asymmetry introduced by the confining potential, a form reminiscent of a quantum-mechanical transition amplitude: the Vandermonde determinants

play the role of Slater determinants describing antisymmetric many-fermion wavefunctions, and the determinantal form of the transition kernel implies that the fermions are non-interacting albeit subject to an unusual single-particle Hamiltonian. Correlations between $V_{\alpha\alpha}$ then imply interactions between the particles.

We now analyse several limiting cases. If V itself is drawn from a unitary random matrix ensemble $\mathcal{P}(V) \propto e^{-\text{Tr} V^2/2X^2}$ (and is thus of rank $r = N$), the integration over $d\mu_{(N)}(\{\mathbf{a}_\kappa\})$ is subsumed into the integration over V . In ‘Cartesian’ coordinates $\prod_\alpha dV_{\alpha\alpha} \prod_{\alpha < \beta} d\text{Re} V_{\alpha\beta} d\text{Im} V_{\alpha\beta}$ the matrix elements $V_{\alpha\alpha}$ are explicitly uncorrelated and independently Gaussian distributed. The transition kernel is thus proportional to $\det_{\alpha\beta} e^{-X^2(\epsilon_\alpha - \epsilon'_\beta)^2/2}$, corresponding to free propagation of the ordinary one-dimensional fermions in the quantum-mechanical analogy (with X^2 playing the role of imaginary time). An identical joint distribution arises also in the study of one-dimensional non-intersecting random walkers [19].

Returning to the case of *fixed* V , and still considering $r \sim N$, we note that the $N^2 - N$ ‘gauge degrees of freedom’ of V enter only through N combinations $V_{\alpha\alpha}$. An application of the central limit theorem then implies that, after integrating over the ‘gauge degrees of freedom’, $V_{\alpha\alpha}$ become independently distributed with the Gaussian weight $\exp\{-N^2 V_{\alpha\alpha}^2/2 \text{Tr} V^2\}$. Consequently, the form of the kernel coincides with the case when V is random.

Let us turn now to the case when $r \sim O(1)$. The remaining step in the derivation of the joint distribution function of the eigenvalues is to implement the integration over $d\mu_{(r)}(\{\mathbf{a}_\kappa\})$. This can be done in two ways which lead to different forms of the distribution functions but are ultimately equivalent in the $N \rightarrow \infty$ limit when applied to the calculation of the correlation functions. The first (exact) procedure requires enforcing the conditions of orthonormality between the r complex vectors \mathbf{a}_κ . Although quite simple in the $r = 1$ case, already for $r = 2$ it leads to a somewhat lengthy calculation, as we will show below, and it quickly becomes unmanageable for larger values of r . If, however, the ultimate interest is in the correlation functions R_{nm} in which the scaling limit $N \rightarrow \infty$ is taken at *fixed* values of n and m , integrating over the remaining $2N - n - m$ ‘energies’ leaves the correlation function dependent only on a finite number of components $a_{\kappa\alpha}$ of the r vectors \mathbf{a}_κ . A well-known result in RMT [2] states that the latter are distributed independently according to the Porter–Thomas formula $e^{-N|a_{\kappa\alpha}|^2}$. The distribution functions obtained by replacing the *whole* of $d\mu_{(r)}(\{\mathbf{a}_\kappa\})$ with the product of independent Porter–Thomas distributions for each of the Nr components of \mathbf{a}_κ can be interpreted as resulting from the substitution of a ‘canonical’ ensemble for the gas of levels in place of the ‘micro-canonical’ ensemble implied by the identity $\text{Tr} H' = \text{Tr}(H + V)$.

2.1. Exact (‘micro-canonical’) joint distribution functions

To illustrate the method, let us first consider the $r = 1$ case. The corresponding distribution function and the lowest order nontrivial correlation function (R_{11} in our notation) have been derived using a somewhat different approach in [21]. V is parametrized as $vN\mathbf{a} \otimes \mathbf{a}^\dagger$, and for a single normalized complex vector \mathbf{a} the measure $d\mu_{(1)}(\mathbf{a})$ becomes

$$d\mu_{(1)}(\mathbf{a}) = J_1 \delta(\mathbf{a}^\dagger \mathbf{a} - 1) e^{-\eta \mathbf{a}^\dagger \mathbf{a}} d[\mathbf{a}] \quad (17)$$

where $d[\mathbf{a}]$ denotes the unconstrained measure $\prod_{\alpha=1}^N da_\alpha da_\alpha^*/\pi$, and η is a positive infinitesimal introduced for convergence. The normalization constant J_1 is determined from

$$1 = J_1 \int d[\mathbf{a}] \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ix - i(x-i\eta)\mathbf{a}^\dagger \mathbf{a}} = J_1 \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{e^{ix}}{(ix + \eta)^N} = \frac{J_1}{(N-1)!}. \quad (18)$$

Substituting this measure into (16) we obtain

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) &= \frac{C_U |c|^2 \mu_N}{(2\pi)^{N^2-N}} \exp\left(-\sum_{\alpha=1}^N W(\epsilon_\alpha)\right) \Delta(H)\Delta(H') \frac{(N-1)!}{(N|v|)^{N-1}} \\ &\times \delta\left(\sum_{\beta} \epsilon'_\beta - \sum_{\alpha} \epsilon_\alpha - Nv\right) \det_{\alpha\beta} \theta[\text{Sign } v(\epsilon'_\beta - \epsilon_\alpha)]. \end{aligned} \tag{19}$$

As expected [21], the determinant is nonzero only if the two sets of levels satisfy the ‘interleaved comb’ constraint $\epsilon_\alpha < \epsilon'_\alpha < \epsilon_{\alpha+1}$ for $v > 0$ and $\epsilon_{\alpha-1} < \epsilon'_\alpha < \epsilon_\alpha$ for $v < 0$.

We next consider the $r = 2$ case. It is convenient to use a symmetrized expression $V = vN(\mathbf{a} \otimes \mathbf{a}^\dagger - \mathbf{b} \otimes \mathbf{b}^\dagger)$ where \mathbf{a} and \mathbf{b} are two complex mutually orthogonal N -component vectors of length 1, and v is the scalar (positive) parameter characterizing the magnitude of V . More general expressions can be reduced to this form by a simple rescaling in the final results. It is also convenient to re-exponentiate the δ -functions in the determinant in (16) so that the λ integrals are performed at a later stage in the calculation. The measure $d\mu_{(2)}(\mathbf{a}, \mathbf{b})$ is explicitly parametrized as

$$d\mu_{(2)}(\mathbf{a}, \mathbf{b}) = J_2 d[\mathbf{a}] d[\mathbf{b}] \delta(\mathbf{a}, \mathbf{b}) e^{-\eta(\mathbf{a}^\dagger \mathbf{a} + \mathbf{b}^\dagger \mathbf{b})} \tag{20}$$

where $\delta(\mathbf{a}, \mathbf{b})$ stands for

$$\delta(\mathbf{a}^\dagger \mathbf{a} - 1) \delta(\mathbf{b}^\dagger \mathbf{b} - 1) \delta(\mathbf{a}^\dagger \mathbf{b} + \mathbf{b}^\dagger \mathbf{a}) \delta\left(\frac{1}{i}[\mathbf{a}^\dagger \mathbf{b} - \mathbf{b}^\dagger \mathbf{a}]\right) \tag{21}$$

and J_2 is the normalization constant. To compute J_2 , the δ -functions are exponentiated with the help of two real Lagrange multipliers x and y , and a complex Lagrange multiplier z :

$$\begin{aligned} 1 &= J_2 \int [d\mathbf{a}][d\mathbf{b}] \delta(\mathbf{a}, \mathbf{b}) \exp\{-\eta(\mathbf{a}^\dagger \mathbf{a} + \mathbf{b}^\dagger \mathbf{b})\} = J_2 \int \frac{dx dy d\text{Re } z d\text{Im } z}{(2\pi)^4} [d\mathbf{a}][d\mathbf{b}] \\ &\times \exp\{i(x - i\eta)(1 - \mathbf{a}^\dagger \mathbf{a}) + i(y - i\eta)(1 - \mathbf{b}^\dagger \mathbf{b}) - iz\mathbf{a}^\dagger \mathbf{b} - iz^* \mathbf{b}^\dagger \mathbf{a}\}. \end{aligned}$$

Performing the integrals, we find

$$J_2 = 4\pi \left(\frac{2}{\pi}\right)^{2N} (-1)^N (N-1)!(N-2)!. \tag{22}$$

With these definitions, we have

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) &= \frac{C_U J_2}{(2\pi)^{N^2}} \Delta(H)\Delta(H') \int d\mu(U_\lambda) \prod_{k=1}^N d\lambda_k d[\mathbf{a}] d[\mathbf{b}] \delta(\mathbf{a}, \mathbf{b}) \exp\{i \text{Tr}(\lambda V)\} \\ &\times \sum_{\sigma, \sigma'} (-1)^{\sigma+\sigma'} \exp\left\{i \sum_{\alpha=1}^N (\epsilon_\alpha \lambda_{\sigma(\alpha)} - \epsilon'_\alpha \lambda_{\sigma'(\alpha)})\right\} \exp\left\{-\sum_{\alpha=1}^N W(\epsilon_\alpha)\right\}. \end{aligned} \tag{23}$$

Proceeding as in the computation of J_2 , we obtain

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) &= \frac{C_U |J_2| |c|^2 \mu_N}{(2\pi)^{N^2}} \Delta(H)\Delta(H') \left(\frac{\pi^2}{Nv}\right)^N \int \frac{dx dy d\text{Re } z d\text{Im } z}{(2\pi)^4} e^{i(x+y)} \\ &\times \sum_{\sigma, \sigma'} (-1)^{\sigma+\sigma'} \prod_{k=1}^N \int d\lambda_k \frac{\exp\{i\lambda_k(\epsilon_{\sigma(k)} - \epsilon'_{\sigma'(k)})/Nv\}}{(\lambda_k - x + i\eta)(\lambda_k + y - i\eta) + |z|^2} \\ &\times \exp\left\{-\sum_{\alpha=1}^N W(\epsilon_\alpha)\right\}. \end{aligned} \tag{24}$$

After integrating over the eigenvalues of λ we obtain

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) &= \frac{C_U |J_2| |c|^2 \mu_N N!}{(2\pi)^{N^2-N}} \Delta(H) \Delta(H') \left(\frac{\pi^2}{Nv}\right)^N \frac{1}{8(2\pi)^3} \exp\left(-\sum_{\alpha=1}^N W(\epsilon_\alpha)\right) \\ &\times \sum_{\sigma} (-1)^\sigma \int_{-\infty}^{\infty} d\tau \exp\left\{\frac{i\tau \sum_{\alpha=1}^N (\epsilon_\alpha - \epsilon'_{\sigma(\alpha)})}{2Nv}\right\} \int_{-\infty}^{\infty} dt e^{it} \int_0^{\infty} \zeta d\zeta \\ &\times \left\{ \theta(t^2 - \zeta^2) (-is_t)^N \frac{\exp\{-i\sqrt{t^2 - \zeta^2} s_t \Delta_\sigma\}}{(\sqrt{t^2 - \zeta^2} - i\eta s_t)^N} \right. \\ &\left. + \theta(\zeta^2 - t^2) \frac{\exp\{-\sqrt{\zeta^2 - t^2} \Delta_\sigma\}}{(\sqrt{\zeta^2 - t^2} + i\eta s_t)^N} \right\} \end{aligned}$$

where $\tau = x - y$, $t = x + y$, $\zeta = 2|z|$, $s_t \equiv \text{Sign } t$, the ambiguities in the analytic properties of the integrand are resolved by letting $t \rightarrow t - i\eta$, and $\Delta_\sigma = \sum_{\alpha=1}^N |\epsilon_\alpha - \epsilon'_{\sigma(\alpha)}|/2Nv$. The extra factor $N!$ comes from performing one of the sums over permutations. Note that the parameter Δ_σ depends only on the absolute values of the distances between the levels in contrast to the case of rank $r = 1$ perturbation [21].

Integration over τ gives a factor of $4\pi Nv \delta[\sum_{\alpha} (\epsilon_\alpha - \epsilon'_\alpha)]$ which simply enforces the identity $\text{Tr } H = \text{Tr } H'$. Performing the remaining integrations, we find

$$\begin{aligned} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) &= \frac{2C_U |c|^2 \mu_N}{(2\pi)^{N^2-N}} \left(\frac{4}{Nv}\right)^{N-1} N!(N-1)! \delta\left[\sum_{\alpha} (\epsilon_\alpha - \epsilon'_\alpha)\right] \\ &\times \Delta(H) \Delta(H') \exp\left(-\sum_{\alpha=1}^N W(\epsilon_\alpha)\right) \sum_{\sigma} (-1)^\sigma \theta(1 - \Delta_\sigma) (1 - \Delta_\sigma)^{N-2}. \quad (25) \end{aligned}$$

Note the hard constraint on the sum of the absolute values of the level shifts $\sum_{\alpha} |\epsilon_\alpha - \epsilon'_{\sigma(\alpha)}| \leq 2Nv$. Note also that this result cannot be simplified in an obvious way in the $N \rightarrow \infty$ limit since Δ_σ does not scale as $1/N$. Indeed, the natural scale for v is the mean level spacing $\bar{\Delta}$, and $\sum_{\alpha} |\epsilon_\alpha - \epsilon'_{\sigma(\alpha)}| \sim O(N\bar{\Delta})$ so that the $\Delta_\sigma \sim O(1)$. The distribution functions (16) and (27) below are equivalent only in the sense that they lead to identical correlation functions in the ‘thermodynamic’ limit $N \rightarrow \infty$.

2.2. ‘Canonical’ joint distributions

As discussed above, after neglecting $O(1/N)$ corrections, the measure $d\mu_{(r)}$ for finite r is brought to the simple form

$$d\mu_{(r)}(\{\mathbf{a}_\kappa\}) = \prod_{\kappa=1}^r \prod_{\alpha=1}^N \frac{N da_{\kappa\alpha} da_{\kappa\alpha}^*}{\pi} e^{-N|a_{\kappa\alpha}|^2}. \quad (26)$$

Due to the factorized structure of this measure, the transition kernel in (16) acquires the determinantal form

$$\det_{\alpha\beta} \hat{\mathcal{D}}_{\{\epsilon; \hat{v}\}}^{-1} \delta(\epsilon_\alpha - \epsilon_\beta) \quad (27)$$

where the action of the differential operator $\hat{\mathcal{D}}_{\{\epsilon; \hat{v}\}}$ on an arbitrary function $f(\epsilon)$ is defined as

$$\hat{\mathcal{D}}_{\{\epsilon; \hat{v}\}} f(\epsilon) = \det(\hat{1} - \hat{v} d/d\epsilon) f(\epsilon) \quad (28)$$

and its inverse has a convenient integral representation

$$\hat{D}_{\{\epsilon; \hat{v}\}}^{-1} f(\epsilon) = \int \prod_{\kappa=1}^r \frac{d\psi_{\kappa} d\psi_{\kappa}^*}{\pi} e^{-|\psi|^2} f(\epsilon + \psi^* \hat{v} \psi). \tag{29}$$

In the opposite case of infinite rank, the central limit theorem ensures that the diagonal matrix elements $V_{\alpha\alpha}$ are independently Gaussian distributed. In consequence, the differential operator $\hat{D}_{\{\epsilon; \hat{v}\}}$ simplifies to

$$\hat{D}_{\{\epsilon; \hat{v}\}} = \exp \left\{ -\frac{1}{2} \text{Tr} \hat{v}^2 \frac{d^2}{d\epsilon^2} \right\}. \tag{30}$$

Here and below when considering the infinite rank case we assume $\text{Tr} V = 0$ since a nonzero value of $\text{Tr} V$ can be accounted for by a trivial uniform shift of the spectrum.

3. Parametric correlation functions

3.1. Parametric correlation functions in the energy representation

The objective of this section is to present a derivation of the multipoint correlation function R_{nm} (equations (38) and (44) below). The joint distribution function defined by equations (16) and (27) serves as the basis for this calculation which employs a scheme based on the orthogonal polynomial expansion. In accordance with the definition (7), the parametric correlation function is represented as

$$R_{nm} (\{E_a\}_{a=1}^n, \{E'_b\}_{b=1}^m) = \int \prod_{\alpha=1}^N d\epsilon_{\alpha} \prod_{\beta=1}^N d\epsilon'_{\beta} \mathcal{P}(\{\epsilon\}, \{\epsilon'\}) S(\{E\}) S'(\{E'\}) \tag{31}$$

where the source terms $S(\{E\})$ and $S'(\{E'\})$ are defined as

$$S(\{E\}) = \prod_{a=1}^n \sum_{\alpha_a=1}^N \delta(E_a - \epsilon_{\alpha_a}) \quad S'(\{E'\}) = \prod_{b=1}^m \sum_{\beta_b=1}^N \delta(E'_b - \epsilon'_{\beta_b}). \tag{32}$$

Integrating over all ϵ' we obtain

$$R_{nm} (\{E_a\}_{a=1}^n, \{E'_b\}_{b=1}^m) = \int \prod_{\alpha=1}^N d\epsilon_{\alpha} \Delta(\epsilon) \exp \left\{ -\sum_{\alpha=1}^N W(\epsilon_{\alpha}) \right\} S(\{E\}) \times \prod_{\beta=1}^N \hat{D}_{\{\epsilon_{\beta}; \hat{v}\}}^{-1} \left[\Delta(\epsilon) \prod_{b=1}^m \sum_{\beta_b=1}^N \delta(E'_b - \epsilon_{\beta_b}) \right]. \tag{33}$$

Let us denote as π_n and π_m , respectively, the set of all *distinct* indices α_a in the multiple sum over $\alpha_1, \dots, \alpha_n$, and the set of all distinct indices β_b in the multiple sum over β_1, \dots, β_m . It is now convenient to split the product of \hat{D}^{-1} as

$$\prod_{\beta \in \pi_m} \hat{D}_{\{\epsilon_{\beta}; \hat{v}\}}^{-1} \prod_{\beta=1, \beta \notin \pi_m}^N \hat{D}_{\{\epsilon_{\beta}; \hat{v}\}}^{-1}. \tag{34}$$

All the factors in the second product act only on the variables in $\Delta(\epsilon)$ while the operators in the first product act on the variables in both Δ and the source term δ -functions. The Vandermonde determinant involves only polynomials of degree $p - 1$ in its p th row, so that the action of a

product of N identical differential operators transforms the elements in its p th row into different polynomials of the same degree and with the same leading term. Consequently,

$$\prod_{\beta=1}^N \hat{\mathcal{D}}_{\{\epsilon_\beta; \hat{v}\}}^{-1} \Delta(\epsilon) \equiv \Delta(\epsilon) \quad \text{and} \quad \prod_{\beta=1, \beta \notin \pi_m}^N \hat{\mathcal{D}}_{\{\epsilon_\beta; \hat{v}\}}^{-1} \Delta(\epsilon) = \prod_{\beta \in \pi_m} \hat{\mathcal{D}}_{\{\epsilon_\beta; \hat{v}\}} \Delta(\epsilon). \tag{35}$$

Integrating by parts over the variables in π_m we arrive at

$$R_{nm}(\{E_a\}_{a=1}^n, \{E'_b\}_{b=1}^m) = \sum_{\substack{\beta_1, \dots, \beta_m \\ \alpha_1, \dots, \alpha_n}} \int \prod_{\alpha=1}^N d\epsilon_\alpha \left[\prod_{\beta \in \pi_m} \hat{\mathcal{D}}_{\{\epsilon_\beta; \hat{v}\}} \Delta(\epsilon) \right] \prod_{b=1}^m \delta(E'_b - \epsilon_{\beta_b}) \\ \times \prod_{\beta \in \pi_m} \hat{\mathcal{D}}_{\{\epsilon_\beta; \hat{v}\}}^{\dagger-1} \left[\exp \left\{ - \sum_{\alpha=1}^N W(\epsilon_\alpha) \right\} \Delta(\epsilon) \prod_{a=1}^n \delta(E_a - \epsilon_{\alpha_a}) \right] \tag{36}$$

where $\hat{\mathcal{D}}_{\{\epsilon; \hat{v}\}}^\dagger = \hat{\mathcal{D}}_{\{\epsilon; -\hat{v}\}}$. The above expression is manifestly asymmetric with respect to the confining potential $W(\epsilon)$. As long as all the energies are close to the centre of the band, and the change in the average DoS induced by the potential V is negligible, the action of $\hat{\mathcal{D}}$ on W can be ignored. It is instructive, however, to consider a more general case. To this end, we introduce a new operator

$$\hat{\hat{\mathcal{D}}}_{\{\epsilon; \hat{v}\}} = \exp\{-W(\epsilon)/2\} \hat{\mathcal{D}}_{\{\epsilon; \hat{v}\}} \exp\{W(\epsilon)/2\}. \tag{37}$$

When (36) is rewritten in terms of $\hat{\hat{\mathcal{D}}}$ and $\hat{\hat{\mathcal{D}}}^\dagger$, each of the Vandermonde determinants appears in combination with $\prod_{\alpha=1}^N \exp\{-W(\epsilon_\alpha)/2\}$, so that they can be equivalently written as $\det_{\alpha\beta} \varphi_{\alpha-1}(\epsilon_\beta)$ where $\varphi_{\alpha-1}(\epsilon) = \exp\{-W(\epsilon)/2\} p_{\alpha-1}(\epsilon)$, and $p_\alpha(\epsilon)$ are the polynomials orthogonal with respect to the weight $\exp\{-W(\epsilon)\}$. Expanding the determinants, and using the orthonormality of the set of functions $\varphi_\alpha(\epsilon)$, we arrive at the following representation of the correlation function as an $(n+m) \times (n+m)$ determinant,

$$R_{nm}(\{E_a\}_{a=1}^n, \{E'_b\}_{b=1}^m) \\ = \det \begin{pmatrix} k(E_{a_1}, E_{a_2}) & \hat{\hat{\mathcal{D}}}_{\{E'_{b_2}; \hat{v}\}}^{-1} [k(E_{a_1}, E'_{b_2}) - \delta(E_{a_1} - E'_{b_2})] \\ \hat{\hat{\mathcal{D}}}_{\{E'_{b_1}; \hat{v}\}}^\dagger k(E'_{b_1}, E_{a_2}) & \hat{\hat{\mathcal{D}}}_{\{E'_{b_2}; \hat{v}\}}^{-1} \hat{\hat{\mathcal{D}}}_{\{E'_{b_1}; \hat{v}\}}^\dagger k(E'_{b_1}, E'_{b_2}) \end{pmatrix} \tag{38}$$

where the indices a and a' (b and b') run from 1 to n (from 1 to m). The kernel

$$k(E, E') = \sum_{p=0}^{N-1} \varphi_p(E) \varphi_p(E') \tag{39}$$

describes the correlations among the levels of the original matrices H , while the lower right corner of the matrix describes the correlation functions *within* the ensemble defined by the distribution function $\mathcal{P}(H) = C_U \exp\{-\text{Tr } W(H - V)\}$. The cross-correlations between H and $H' = H + V$ are encoded in the off-diagonal blocks of (38). In the bulk scaling limit the kernel k takes the form [11]

$$k(E, E') = \text{Im} \exp\{i\pi\omega\rho(\epsilon)\}/\omega \equiv \pi\rho(\epsilon)k[\pi\rho(\epsilon)\omega] \tag{40}$$

where $\omega = E' - E$ and $\epsilon = (E' + E)/2$.

The bulk of the present paper is devoted to the case when the strength of the perturbation V is such that the parametric correlators between the ‘old’ and the ‘new’ levels are $O(1)$. This condition is fulfilled if $\text{Tr } V^2 \sim O(N^2 \Delta^2)$. In this regime, the operator $\hat{\hat{\mathcal{D}}}_{\{\epsilon; \hat{v}\}}$ can be

explicitly evaluated using a representation of \hat{D} as an integral over a vector of Grassmann variables χ ,

$$\begin{aligned} \hat{D}_{\{\epsilon; \hat{v}\}} &\equiv \exp[-W(\epsilon)/2] \int d\chi d\chi^* \exp\left\{-|\chi|^2 - \chi^* \hat{v} \chi \frac{d}{d\epsilon}\right\} \exp[W(\epsilon)/2] \\ &\approx \int d\chi d\chi^* \exp[W'(\epsilon) \chi^* \hat{v} \chi / 2] \exp\left\{-|\chi|^2 - \chi^* \hat{v} \chi \frac{d}{d\epsilon}\right\} \\ &= \det[1 - \hat{v} W'(\epsilon)/2] \hat{D}_{\{\epsilon; \hat{t}(\hat{v})\}} \end{aligned} \tag{41}$$

where $\hat{t}(\hat{v}) = \hat{v}/(1 - \hat{v} W'(\epsilon)/2)$. Assuming that all the energies in both sets $\{E\}$ and $\{E'\}$ lie within a few level spacings of some central energy ϵ , and since the determinantal structure of (38) ensures that the operators \hat{D} and \hat{D}^{-1} come in pairs, the determinant in the prefactor can be dropped.

The derivative of the confining potential is related to the real part of the *average* Green function via

$$W'(\epsilon) = 2 \int \frac{\rho(z)}{z - \epsilon} \equiv 2 \operatorname{Re} G(\epsilon). \tag{42}$$

Consequently, $\tau(v)$ can be represented as the solution of the equation

$$\hat{t}(\hat{v}) = \hat{v} + \hat{v} \operatorname{Re} G(\epsilon) \hat{t}(\hat{v}) \tag{43}$$

which can be immediately identified as the equation for the *reactance matrix* for scattering off the potential V . In fact, as can be shown using the methods of statistical field theory [12], the parametrization of \hat{D} in terms of the corresponding reactance matrix retains its validity in the more general case of *non-invariant* distributions $\mathcal{P}(H)$. The eigenvalues of the reactance matrix are conveniently parametrized as $\tan \delta_\kappa$ where δ_κ are the corresponding phase shifts.

As discussed above, the parametric correlations generically fall into two regimes corresponding to the rank of V being finite or infinite. In the latter case, the condition $\operatorname{Tr} V^2 \sim O(N^2 \Delta^2)$ ensures that, for a generic eigenvalue v_i of \hat{v} , and at a generic position ϵ in the spectrum, $v_i W'(\epsilon) \ll 1$. Thus the distinction between \hat{D} and \hat{D} is meaningful only in the finite rank case, and to simplify notation we will use the symbol \hat{D} in both cases with its parametrization implied by the context.

Similarly, to the leading order in the $1/N$ expansion, the DoS $\rho(\epsilon)$ can be approximated by an ϵ -independent inverse mean level spacing $1/\bar{\Delta}$, thus simplifying (38) to the form given in [17],

$$\begin{aligned} \mathfrak{R}_{nm}(\{E_a\}_{a=1}^n, \{E'_b\}_{b=1}^m) \\ = \det \begin{pmatrix} k(\epsilon_{a_2} - \epsilon_{a_1}) & \hat{D}_{\{\epsilon'_{b_2}; \hat{v}\}}^{-1} (k[\epsilon'_{b_2} - \epsilon_{a_1}] - \delta[\epsilon'_{b_2} - \epsilon_{a_1}]) \\ \hat{D}_{\{\epsilon_{a_2}; \hat{v}\}} k(\epsilon_{a_2} - \epsilon'_{b_1}) & k(\epsilon'_{b_2} - \epsilon'_{b_1}) \end{pmatrix} \end{aligned} \tag{44}$$

where $\hat{v} = \hat{v}/\bar{\Delta}$ and $\mathfrak{R}_{nm} = \bar{\Delta}^{n+m} R_{nm}$. In the $r \rightarrow \infty$ limit this equation reproduces the results obtained previously [7, 18]. The representation of R_{nm} in the form of λ integrals characteristic of the non-linear sigma model calculations [7, 12] corresponds to the Fourier decomposition of $\hat{D}k$ and $\hat{D}^{-1}[k - \delta]$ in (44).

3.2. Universality in shifted distributions

To draw a connection with some previous studies of parametrically deformed random matrix ensembles [22, 23], let us briefly consider the opposite case of a perturbation strong enough to significantly affect the DoS. In this regime, the cross-correlations are small as $1/N$. There remains, however, the question of whether the correlation functions of a matrix ensemble

defined by the shifted distribution $\mathcal{P}_V(H) = C_U \exp\{-\text{Tr } W(H - V)\}$ are universal. To the best of our knowledge, this question has been answered so far in the affirmative only in the case when W is quadratic [22] (see also [24] where the case of non-Gaussian potential with a linear source term was considered). The formalism developed above affords an opportunity to extend the proof to the case of arbitrary $W(H)$ (still, however, assuming strong confinement). The corresponding correlation functions, as follows from (38), have the determinantal structure with the kernel $k_v(\omega, \epsilon) = \hat{D}_{\{\epsilon_1; \hat{v}\}}^{-1\dagger} \hat{D}_{\{\epsilon_2; \hat{v}\}} k(\omega, \epsilon)$. The operator \hat{D} in this regime is given by equation (30). Expanding the difference $W(\epsilon_2) - W(\epsilon_1) \approx \omega W'(\epsilon)$, we obtain

$$k_v(\omega, \epsilon) = \exp\{-\omega W'(\epsilon)/2\} \exp\{-\pi^2 x^2 \partial_\omega \partial_\epsilon\} \exp\{\omega W'(\epsilon)/2\} \text{Im} \frac{\exp\{i\omega\pi\rho(\epsilon)\}}{\pi\omega} \tag{45}$$

where $x = (\text{Tr } \hat{v}^2)^{1/2}/\pi$. By virtue of (42), the last two exponents can be combined to obtain $\text{Im} \exp\{\omega\mathcal{G}(\epsilon)\}$, where $\mathcal{G}(z)$ is defined as the analytic continuation of the retarded Green function $G(\epsilon) = \int dz \rho(z)/(z - \epsilon) + i\pi\rho(\epsilon)$ onto the complex plane with cuts along the real axis *outside* the support of the DoS $\rho(\epsilon)$. By employing an integral representation of the differential operator

$$\exp\{-\pi^2 x^2 \partial_\omega \partial_\epsilon\} = \int \frac{d\zeta d\zeta^*}{\pi^3 x^2} \exp\{-|\zeta|^2/\pi^2 x^2 - \zeta^* \partial_\omega + \zeta \partial_\epsilon\} \tag{46}$$

and differentiating with respect to ϵ , equation (45) is brought to the form

$$\begin{aligned} &\partial_\epsilon [\exp\{\omega W'(\epsilon)/2\} k_v(\omega, \epsilon)] \\ &= \int \frac{d\zeta d\zeta^*}{\pi^3 x^2} \exp\{-|\zeta|^2/\pi^2 x^2\} \text{Im} \exp\{(\omega - \zeta^*)\mathcal{G}(\epsilon + \zeta)\} \mathcal{G}'(\epsilon + \zeta). \end{aligned} \tag{47}$$

After a change of variables $r = |\zeta|^2, u = e^{i\arg\zeta}$ we note that, since \mathcal{G} is analytic by construction, the only non-analyticity of the integral over u is due to the pole generated by the solution $u_0(\epsilon)$ of the equation $\pi^2 x^2 \mathcal{G}(\epsilon + u) + u = 0$ (existence of multiple solutions of this equation would contradict level number conservation). We thus find, upon rescaling u and integrating over r ,

$$\begin{aligned} \partial_\epsilon [\exp\{\omega W'(\epsilon)/2\} k_v(\omega, \epsilon)] &= \text{Im} \oint du \frac{\exp\{\omega\mathcal{G}(\epsilon + u)\}}{\pi^2 x^2 \mathcal{G}(\epsilon + u) + u} \mathcal{G}'(\epsilon + u) \\ &= \text{Im} \frac{\exp\{-\omega u_0(\epsilon)/\pi^2 x^2\} \mathcal{G}'(\epsilon + u_0(\epsilon))}{1 + \pi^2 x^2 \mathcal{G}'(\epsilon + u_0(\epsilon))} \end{aligned} \tag{48}$$

where the contribution from integrals along the cuts vanishes upon taking the imaginary part of this expression. Integrating over ϵ and using $[1 + \pi^2 x^2 \mathcal{G}'(\epsilon + u_0(\epsilon))](du_0/d\epsilon) = -\pi^2 x^2 \mathcal{G}'(\epsilon + u_0(\epsilon))$, we find

$$k_v(\omega, \epsilon) = \exp\{-\omega W'(\epsilon)/2 + \omega \text{Re } u_0(\epsilon)\} \frac{\sin \omega \rho_v(\epsilon)}{\omega} \tag{49}$$

where the renormalized DoS is $\rho_v(\epsilon) = -\text{Im } u_0(\epsilon)/\pi^2 x^2$. The exponential prefactors cancel out due to the determinantal structure of the correlation functions, thus establishing their universal form.

3.3. Parametric correlation functions in the number representation

The ‘complementary’ number-dependent correlation functions are generically expressed as derivatives of the corresponding generating functions. The archetypal generating function is the probability $\mathfrak{P}_{nn'}(J, J')$ to find $n(J) = n$ levels in the interval J of the unperturbed sequence and n' levels in the interval J' of the perturbed sequence, where $n(J)$ is the number

of levels in the interval J :

$$\mathfrak{P}_{nn'}(J, J'; \mathfrak{r}) \equiv \langle \delta_{n,n(J)} \delta_{n',n(J')} \rangle = \frac{(-1)^{n+n'}}{n!n'} \sum_{k=n}^{\infty} \sum_{k'=n'}^{\infty} \frac{(-1)^{k+k'} r_{kk'}}{(k-n)!(k'-n)!}. \tag{50}$$

Here $r_{kk'}$ represents the correlation function $R_{kk'}$ integrated over the direct product of intervals $J^{\otimes k} \otimes J'^{\otimes k'}$ with the corresponding measures dm_J and $dm_{J'}$. The intervals J and J' may, generically, consist of an arbitrary number of disjointed segments. Equation (50) is a straightforward generalization of the non-parametric analogue $P_n(J)$ (see, e.g., appendix 7 of [2]). For later convenience in carrying out the summations, $r_{kk'}$ can be represented in the form of a fermionic functional integral

$$r_{kk'} = \det K \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \left(\int dm_J(u) \bar{\xi}(u) \xi(u) \right)^k \left(\int dm_{J'}(w) \bar{\eta}(w) \eta(w) \right)^{k'} \times \exp \left\{ \int du \int dw \bar{\Psi}(u) K^{-1}(u, w) \Psi(w) \right\} \tag{51}$$

where $\bar{\Psi} = (\bar{\xi}, \bar{\eta})$ is a fermionic doublet. The matrix kernel \hat{K} is easily reconstructed from (38),

$$K(u, w) = \begin{pmatrix} k(u, w) & \hat{D}_w^{-1}[k(u, w) - \mathbb{I}] \\ \hat{D}_u^\dagger k(u, w) & \hat{D}_w^{-1} \hat{D}_u^\dagger k(u, w) \end{pmatrix}. \tag{52}$$

Here the matrix elements of the scalar kernel \hat{k} are given by (39), and \mathbb{I} is adopted as a shorthand for the Dirac δ -function. For brevity we also drop the subscript $\hat{t}(\hat{v})$, and \hat{D} is understood to be the function of either the single parameter $\text{Tr } \hat{v}^2$ or the r phase shifts δ_x parametrizing $\hat{t}(\hat{v})$ in the infinite and finite rank cases, respectively.

Equations (50)–(52) can be used directly to compute a certain class of correlation functions exemplified by the correlation between level spacings of the original and perturbed level sequences centred at given *energies* E and E' , respectively. Since our main interest, however, is in the correlation functions in the number representation, we need to be able to relate the levels ϵ'_β in the new sequence to the parametric ‘descendant’ of a given level ϵ_α in the unperturbed sequence. The absence of level crossings allows one to do this by demanding, e.g., that if an interval $J_\epsilon = (-\infty, \epsilon]$ of the unperturbed sequence contains exactly n levels (as counted from the bottom of the band), the corresponding interval $J'_{\epsilon+\omega}$ of the perturbed sequence contains exactly $n + q$ levels, where q is fixed and n is arbitrary. By construction, then, the uppermost level in the interval $J'_{\epsilon+\omega}$ is separated by $q - 1$ levels from the parametric descendant of the uppermost level in the interval J_ϵ . Summing over all n we find the corresponding probability as

$$\mathfrak{P}_q(J_\epsilon, J'_{\epsilon+\omega}; \mathfrak{r}) \equiv \langle \delta_{n(J_\epsilon), n(J'_{\epsilon+\omega})+q} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!(n+q)!} \sum_{\substack{k=n \\ k'=n+q}}^{\infty} \frac{(-1)^{k+k'+q} r_{kk'}}{(k-n)!(k'-n-q)!} \\ = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{iq\phi} \sum_{\substack{k=0 \\ k'=q}}^{\infty} \frac{(-1)^{k+k'+q} r_{kk'}}{k!k'} z_+^k z_-^{k'} \tag{53}$$

where $z_\pm = 1 + e^{\pm i\phi}$. Substituting (51) into (53), one finds that

$$\mathfrak{P}_q(J, J'; \mathfrak{r}) = (-1)^q \int_0^{2\pi} \frac{d\phi}{2\pi} e^{iq\phi} Z_q(\omega, \mathfrak{r}; \phi) \tag{54}$$

where

$$Z_q(\omega, \mathfrak{r}; \phi) = \det[\mathbb{I}\sigma_0 - \hat{K} \Pi(\phi)] - \sum_{k'=0}^{q-1} \frac{\partial_{\gamma}^{k'}}{k'!} \det[\mathbb{I}\sigma_0 - \hat{K} \Pi_\gamma(\phi)] \Big|_{\gamma=0} \tag{55}$$

and, denoting the Pauli matrices as σ_i ,

$$\Pi(\phi) = \begin{pmatrix} z_+ & 0 \\ 0 & z_- \end{pmatrix} \quad \Pi_\gamma(\phi) = \begin{pmatrix} z_+ & 0 \\ 0 & \gamma z_- \end{pmatrix}.$$

The determinants are understood as functional determinants on the space of two-component functions defined on the product interval $J \otimes J'$.

Although the formalism in principle can be applied at the spectral edge, we are primarily interested in the correlation functions in the bulk of the spectrum. Taking the scaling limit would then greatly facilitate the calculation. In order to avoid unessential complications related to the energy dependence of the average DoS, we replace \hat{k} in (52) with its value (40) in the scaling limit assuming $\bar{\Delta} \equiv \rho^{-1}(\epsilon)$ is a constant, and then regularize k as

$$k_\eta(u - w) = \frac{\sin \pi(u - w)}{\pi(u - w)} \exp[-(1/2)\eta(|u| + |w|)] \tag{56}$$

where the limit $\eta \rightarrow 0$ is implied in all expressions involving this kernel, and the energy variables are rescaled by $\bar{\Delta}$. Using equation (51) it is easily shown that $\langle n(J_\infty) \rangle = \langle n(J'_\infty) \rangle = 2/\eta$, and $\langle [n(J_\infty) - n(J'_\infty)]^2 \rangle \sim O(\eta)$. Thus, although the regularization formally violates the level number conservation, the corresponding error tends to zero in the limit $\eta \rightarrow 0$. At the same time, for any finite value of η the number of levels in the semi-infinite intervals J_ϵ and $J'_{\epsilon+\omega}$ is finite, ensuring the convergence of the integrals over energies at the lower limit. In the following we will suppress the index η .

As written, (54) involves a matrix oscillating integral kernel defined on a product of semi-infinite intervals. However, as we will now show for the case $q = 0$, it can be rewritten in the form which is (i) more amenable to numerical analysis, and (ii) makes the integrability of the kernel (in the sense discussed in [25]) manifest. Without loss of generality, we can set $\epsilon = 0$, and shift the variables so as to define the determinant on the quadrant $(-\infty, 0] \otimes (-\infty, 0]$. The corresponding shift operator is absorbed into the redefinition $\hat{\mathcal{D}} \rightarrow e^{\omega d/du} \hat{\mathcal{D}}$. The term involving the δ -function in the upper right corner of (52) can be separated to reveal the dyadic structure of the remainder,

$$\hat{K} = \begin{pmatrix} \mathbb{I} \\ \hat{\mathcal{D}} \end{pmatrix} \otimes (\hat{k} \quad \hat{\mathcal{D}}^{-1} \hat{k}) - \begin{pmatrix} 0 & \hat{\mathcal{D}}^{-1} \mathbb{I} \\ 0 & 0 \end{pmatrix}$$

where we have also used $\hat{\mathcal{D}}_w^\dagger k(u - w) = \hat{\mathcal{D}}_u k(u - w)$. Now using the identities $(\mathbb{I}\sigma_0 + z_- \hat{\mathcal{D}}^{-1} \mathbb{I}\sigma_+)^{-1} = (\mathbb{I}\sigma_0 - z_- \hat{\mathcal{D}}^{-1} \mathbb{I}\sigma_+)$ and $\det(\mathbb{I}\sigma_0 + z_- \hat{\mathcal{D}}^{-1} \mathbb{I}\sigma_+) = 1$, where $\sigma_+ = (\sigma_1 + i\sigma_2)/2$, one obtains

$$\det[\mathbb{I}\sigma_0 - \hat{K} \Pi(\phi)] = \det[\mathbb{I} - (z_+ \hat{k} - z_+ z_- \hat{k} (\hat{\mathcal{D}}^{-1} \mathbb{I}) \hat{\mathcal{D}} + z_- (\hat{\mathcal{D}}^{-1} \hat{k}) \hat{\mathcal{D}})]. \tag{57}$$

Employing the Fourier representations,

$$\begin{cases} k(u) \\ \delta(u) \end{cases} = \frac{1}{2} \int_{-\infty}^{\infty} d\lambda \begin{cases} \theta(1 - \lambda)\theta(1 + \lambda) \\ 1 \end{cases} e^{i\lambda\pi u}$$

one finds that

$$\begin{aligned} & [z_+ \hat{k} - z_+ z_- \hat{k} (\hat{\mathcal{D}}^{-1} \mathbb{I}) \hat{\mathcal{D}} + z_- (\hat{\mathcal{D}}^{-1} \hat{k}) \hat{\mathcal{D}}](u, w) \\ &= \int_{-1}^1 \frac{d\lambda}{2} e^{i\pi\lambda(w-u)} \left\{ z_+ - \frac{z_+ z_-}{2\pi i} \int_{-\infty}^{\infty} d\mu \frac{e^{i\pi w(\mu-\lambda)} \mathcal{D}^{-1}(\mu)}{\lambda - \mu - i\delta} \hat{\mathcal{D}} + z_- \mathcal{D}^{-1}(\lambda) \hat{\mathcal{D}} \right\} \end{aligned} \tag{58}$$

where in the infinite rank case

$$\mathcal{D}(\lambda) = \exp\{i\pi\omega\lambda + \pi^2 x^2 \lambda^2 / 2\} \tag{59}$$

and in the finite rank case

$$\mathcal{D}(\lambda) = \exp\{i\pi\omega\lambda\} \prod_{\kappa=1}^r [1 - i\lambda \tan \delta_\kappa]. \tag{60}$$

Finally, the cyclic invariance of the determinant and the identity $z_+ + z_- = z_+z_-$ are used to perform the integrals in the μ - ω space, bringing the determinant to the form

$$Z_0(\omega, \mathfrak{r}; \phi) \equiv \det[\mathbb{I}\sigma_0 - \hat{K} \Pi(\phi)] = \det_{[-1,1]} [\mathbb{I} - \mathcal{K}(\phi)] \tag{61}$$

where the matrix elements of the kernel $\mathcal{K}(\phi)$ are

$$\mathcal{K}(\lambda, \mu; \phi) = \frac{1}{4\pi i} \sqrt{\mathcal{D}(\lambda)\mathcal{D}(\mu)} \frac{F(\lambda; \phi) - F(\mu; \phi)}{\lambda - \mu} \tag{62}$$

and

$$F(\lambda; \phi) = \frac{(z_+ + z_-)}{\pi i} \int_{-\infty}^{\infty} d\mu \frac{\mathcal{D}^{-1}(\mu)}{\mu - \lambda} - (z_+ - z_-) \mathcal{D}^{-1}(\lambda). \tag{63}$$

Here the integral is understood in the sense of the Cauchy principal value, and the variables λ and μ in (62) are restricted to the interval $[-1, 1]$. We employ the notation \mathfrak{r} to denote the parametric dependence of Z in both finite and infinite rank cases. A similar, although somewhat more cumbersome, expression is easily reconstructed for the contribution of $\det [\mathbb{I}\sigma_0 - \hat{K} \Pi_\nu(\phi)]$ in the $q \neq 0$ case.

Equations (61)–(63) form the central results of this section. In the remainder, we will consider several applications of these equations to the calculation of number-dependent correlation functions. The integral of the generating function itself $\int d\phi Z_0(\omega, \mathfrak{r}; \phi)/2\pi$ gives the probability $\mathfrak{P}_0(\omega, \mathfrak{r}) \equiv \mathfrak{P}_0(J_\epsilon, J_{\epsilon+\omega})$. In order to obtain the correlation functions P_q and C_q introduced above, we rewrite their definitions in the form

$$P_q(\omega, \mathfrak{r}) = \langle \partial_\epsilon \mathbf{n}(J_\epsilon) \partial_\epsilon \mathbf{n}(J'_{\epsilon+\omega}) \delta_{\mathbf{n}(J_\epsilon), \mathbf{n}(J'_{\epsilon+\omega})+q} \rangle \tag{64}$$

$$C_q(\mathfrak{r}) = \langle \partial_\mathfrak{r} \mathbf{n}(J_\epsilon) \partial_\mathfrak{r} \mathbf{n}(J'_{\epsilon+\omega}) \delta_{\mathbf{n}(J_\epsilon), \mathbf{n}(J'_{\epsilon+\omega})+q} \rangle. \tag{65}$$

Using translational invariance both in the energy and the parameter spaces, rescaling the energies by $\bar{\Delta}$, and rescaling the variables in \mathfrak{r} by $C_{0(0)}^{-1/2}$ [7], we obtain for the rescaled correlation functions p_q and c_q [26]

$$p_q(\omega, \mathfrak{r}) = (-1)^q \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{e^{iq\phi}}{z_+z_-} (-\partial_\omega^2) Z_q(\omega, \mathfrak{r}; \phi) \tag{66}$$

$$c_q(\mathfrak{r}) = (-1)^q \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{e^{iq\phi}}{z_+z_-} (-\partial_\mathfrak{r})^2 Z_q(\omega, \mathfrak{r}; \phi). \tag{67}$$

Let us now consider several special cases. (i) In the infinite rank case \mathfrak{r} reduces to a single parameter $x = (\text{Tr } \hat{\nu}^2)^{1/2}/\pi$. The single level universal velocity autocorrelation function $c(x) \equiv c_0(x)$ is plotted in figure 1. For comparison we also show the results of direct numerical simulation of large random matrices [27]. We also find, based on numerical evaluation of equations (66), (61), (62) and (63) that the function $p(\omega, x) \equiv p_0(\omega, x)$, the distribution of parametric single level shifts, has a Gaussian form $p(\omega, x) = \exp\{-\omega^2/2\sigma(x)\}/\sqrt{2\pi\sigma(x)}$, in accordance with the earlier conjecture [20]. An analytical proof of this statement as well as finding an analytical expression for the width $\sigma(x)$ (shown in the inset in figure 1) are some of the open questions posed by the results of the present investigation.

(ii) In the finite rank case further progress can be made due to a simple pole structure of $\mathcal{D}^{-1}(\lambda)$. Leaving a detailed consideration of the finite rank case to a future publication, we

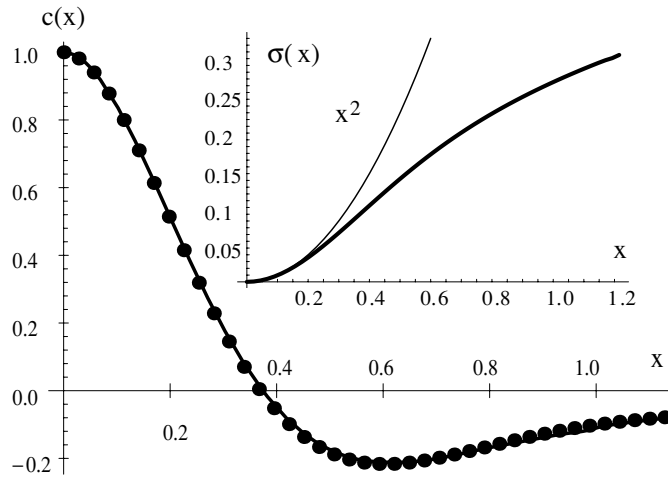


Figure 1. Single level universal velocity autocorrelation function obtained from (67), (61), (62) and (63) (solid line) versus direct numerical simulation of large random matrices [27] (dots). The width $\sigma(x)$ of the Gaussian distribution of level shifts together with the x^2 asymptotics at small x is shown in the inset.

present here the results for c_0 and p_0 in the $r = 1$ case. ζ is now parametrized by a single phase shift δ . In [17] the distribution p_0 for $r = 1$ was found using the fact that, as a special property of $r = 1$, this distribution coincides with the level spacing of the *combined* ensemble of levels $\{\epsilon_\alpha\} \cup \{\epsilon'_\beta\}$. Using the present formalism, this quantity as well as the corresponding c_0 can be evaluated in a more direct way. Introducing an auxiliary *non-parametric* kernel

$$\mathfrak{k}(\lambda, \mu) = \frac{1}{2\pi i} \frac{1 - \exp\{i\pi\omega(\lambda - \mu)\}}{\lambda - \mu} \tag{68}$$

as well as its Green function $g(\lambda, \mu) = (\mathbb{I} - \mathfrak{k})^{-1}$, and the determinant $d(\omega) = \det(\mathbb{I} - \mathfrak{k})$, we find (assuming $\delta > 0$) that

$$c_0^{(r=1)}(\delta) = \frac{d^2}{d\delta^2} \int_0^\infty \frac{d\omega}{2\pi} d(\omega) \exp\{-\pi\omega \arctan \delta\} \int_{-1}^1 d\lambda d\mu \frac{g(\lambda, \mu) \exp\{i\pi\omega\mu\}}{\arctan \delta - i\lambda}. \tag{69}$$

A somewhat more lengthy expression is obtained for p_0 ,

$$p_0^{(r=1)}(\omega; \delta) = \theta(\omega) \frac{\pi d(\omega)}{2} \exp\{-\pi\omega \arctan \delta\} \int_{-1}^1 d\lambda d\mu \frac{g(\lambda, \mu) \exp\{i\pi\omega\mu\}}{1 - i\lambda \tan \delta} \times \left\{ \frac{(1 - i\mu \tan \delta)^2}{\tan \delta} + (1 - i\mu \tan \delta)t(\omega) - T(\omega, \delta) \right\} \tag{70}$$

where $t(\omega) = \int_{-1}^1 d\lambda d\mu g(\lambda, \mu) \exp\{i\pi\omega(\mu - \lambda)\}$, and

$$T(\omega, \delta) = \int_{-1}^1 d\lambda d\mu g(\lambda, \mu) \exp\{i\pi\omega(\mu - \lambda)\}(1 - i\mu \tan \delta). \tag{71}$$

Up to a simple transformation, the kernel \mathfrak{k} is identical to the kernel that controls the *non-parametric* level spacing distribution \mathfrak{S}_1 . In light of the ‘interleaved comb’ constraint on the evolution of levels when $r = 1$, such a connection to \mathfrak{S}_1 is not unexpected.

Within the framework presented here various number-dependent correlation functions in the unitary ensembles can be computed ‘at will’. Nevertheless, several open questions remain

unanswered, and their resolution would greatly enhance our understanding of parametric evolution of random matrices. As already mentioned, the Gaussian form of p_0 in the $r \rightarrow \infty$ case is probably an indicator that the ϕ integration can be performed analytically in some form in the infinite rank case also. If so, a more ‘processed’ expression for $c_0^{(r \rightarrow \infty)}$ may emerge, throwing more light on the properties of the kernel $\mathcal{K}(\phi)$. Separately, an analysis of the asymptotic behaviour of the generating functions along the lines of [28] is a natural way to develop the present formalism further. Finally, the technique of ‘counting’ the number of levels between the spectral edge and a point in the bulk of the spectrum by means of introducing a regularized kernel should be applicable to ensembles with different symmetries, so that if an analogue of (38) is obtained for such ensembles, it is likely to open the door to the computation of the corresponding number-dependent correlation functions.

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